**Exploring Logarithmic Differentiation**

Differentiating functions with variable bases and exponents often presents challenges, particularly when both the base and exponent are functions of xxx. To address this, I use **logarithmic differentiation**, a method that transforms such functions into a form where differentiation is more manageable.

**Logarithmic Differentiation for xf(x)x^{f(x)}xf(x)**

The first scenario I tackled involved differentiating a function of the form xcos⁡(x)x^{\cos(x)}xcos(x), which combines a variable base with a variable exponent. To simplify, I used logarithmic differentiation by taking the natural log of both sides:

y=xcos⁡(x)y = x^{\cos(x)}y=xcos(x)

Applying the logarithm transforms the exponentiation into a product:

ln⁡(y)=cos⁡(x)ln⁡(x)\ln(y) = \cos(x) \ln(x)ln(y)=cos(x)ln(x)

Next, I applied implicit differentiation to find dydx\frac{dy}{dx}dxdy​:

1. Differentiate both sides:

1ydydx=−sin⁡(x)ln⁡(x)+cos⁡(x)x\frac{1}{y} \frac{dy}{dx} = -\sin(x) \ln(x) + \frac{\cos(x)}{x}y1​dxdy​=−sin(x)ln(x)+xcos(x)​

1. Multiply through by yyy (substituting back the original expression for yyy):

dydx=(−sin⁡(x)ln⁡(x)+cos⁡(x)x)xcos⁡(x)\frac{dy}{dx} = \left(-\sin(x) \ln(x) + \frac{\cos(x)}{x}\right) x^{\cos(x)}dxdy​=(−sin(x)ln(x)+xcos(x)​)xcos(x)

This process avoids using the chain rule multiple times and makes differentiation far more efficient.

**Logarithmic Differentiation for Products and Quotients**

In the second problem, I encountered a complex function involving products and quotients:

y=(x+1)3(2x−1)2(x2+1)2y = \frac{(x+1)^3 (2x-1)^2}{(x^2 + 1)^2}y=(x2+1)2(x+1)3(2x−1)2​

Instead of directly applying the quotient and product rules, I utilized logarithmic properties:

ln⁡(y)=3ln⁡(x+1)+2ln⁡(2x−1)−2ln⁡(x2+1)\ln(y) = 3\ln(x+1) + 2\ln(2x-1) - 2\ln(x^2+1)ln(y)=3ln(x+1)+2ln(2x−1)−2ln(x2+1)

Taking the derivative:

1ydydx=3x+1+42x−1−4xx2+1\frac{1}{y} \frac{dy}{dx} = \frac{3}{x+1} + \frac{4}{2x-1} - \frac{4x}{x^2+1}y1​dxdy​=x+13​+2x−14​−x2+14x​

Finally, multiplying through by yyy, I substituted the original expression for yyy:

dydx=(3x+1+42x−1−4xx2+1)(x+1)3(2x−1)2(x2+1)2\frac{dy}{dx} = \left(\frac{3}{x+1} + \frac{4}{2x-1} - \frac{4x}{x^2+1}\right) \frac{(x+1)^3 (2x-1)^2}{(x^2 + 1)^2}dxdy​=(x+13​+2x−14​−x2+14x​)(x2+1)2(x+1)3(2x−1)2​

By rewriting the logarithmic form, I avoided unnecessary complications from nested chain and quotient rules.

**Simplifying Square Roots with Logarithmic Differentiation**

The final example involved a square root of a quotient:

y=2x−1x+2y = \sqrt{\frac{2x-1}{x+2}}y=x+22x−1​​

Expressing this in terms of logarithms, I simplified:

ln⁡(y)=12(ln⁡(2x−1)−ln⁡(x+2))\ln(y) = \frac{1}{2} \left(\ln(2x-1) - \ln(x+2)\right)ln(y)=21​(ln(2x−1)−ln(x+2))

Differentiating both sides:

1ydydx=12(12x−1⋅2−1x+2)\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{2x-1} \cdot 2 - \frac{1}{x+2}\right)y1​dxdy​=21​(2x−11​⋅2−x+21​)

Simplifying and substituting back for yyy:

dydx=12(22x−1−1x+2)2x−1x+2\frac{dy}{dx} = \frac{1}{2} \left(\frac{2}{2x-1} - \frac{1}{x+2}\right) \sqrt{\frac{2x-1}{x+2}}dxdy​=21​(2x−12​−x+21​)x+22x−1​​

Again, logarithmic differentiation allowed me to bypass nested chain rules and directly compute the derivative.

**Observations**

Logarithmic differentiation excels in simplifying derivatives of functions with variable bases and exponents, products, quotients, and composite structures. By transforming these functions into logarithmic form, I was able to apply basic differentiation rules effectively, reducing computational complexity and minimizing potential errors.

This exploration reinforced the utility of logarithmic properties, such as the power, product, and quotient rules, in breaking down intricate expressions. As I work on more advanced problems, I can see how this method will continue to save time and effort in solving complex derivatives.